CHAPTER 13

VECTORS

13.1 Vectors in the Plane

PREREQUISITES

1. There are no special prerequisites for this section other than basic high school algebra.

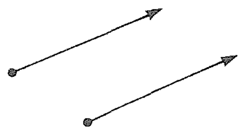
GOALS

1. Be able to perform scalar multiplication, addition, and subtraction with vectors.

STUDY HINTS

1. Adding ordered pairs. The sum of two ordered pairs is another ordered pair. The procedure is simply to sum the corresponding components like this: $(x_1,y_1) + (x_2,y_2) = (x_1 + x_2,y_1 + y_2)$.
2. Scalar multiplication. A scalar is simply a real number, as in Section R.1. Scalar multiplication involves a number times an ordered pair, represented by $r(x,y) = (rx,ry)$. This is opposed to a vector times a vector, which will be discussed later in the chapter.
3. Uniqueness of solution. Example 4 illustrates a general principle: n equations in m unknowns will not have a unique solution if $n < m$.

4. Vectors defined. A vector has both length (magnitude) and direction.



A scalar does not have direction. Two vectors are equal if and only if they both have the same length and the same direction. Pictorially, they do not need to originate from the same starting point. The vectors at the left are equal.

5. Notation. In the text, vectors are usually denoted with boldface letters. In this student guide, vectors are usually denoted with an underlined letter. Sometimes, the vector going from point P to point Q is written as $\overrightarrow{PQ}$. A vector may also be represented by its components (a,b) . Since (a,b) may denote either a vector or a point, you should be careful with your interpretations. Your instructor or other textbooks may use other notations such as a squiggly line (~) underneath a letter or a circumflex (^) over a letter to represent a vector. All of these notations are different, but they all denote a vector.

6. Vector algebra. Addition and scalar multiplication are analogous to the same operations with ordered pairs. Geometrically, $\underline{u} + \underline{v}$ is the diagonal of the parallelogram spanned by $\underline{u}$ and $\underline{v}$. See Fig. 13.1.12. To multiply a vector by r , extend the length of the vector by a factor of r . Reverse the direction if $r < 0$.

7. Vector subtraction. If a sketch like Fig. 13.1.13 is desired, one may forget which direction the difference is pointing. Here is how to remember. If you want $\underline{v} - \underline{w}$, then you should be able to add $\underline{w}$ to obtain $\underline{v}$. If you don't get $\underline{v}$, the direction is not correct.

SOLUTIONS TO EVERY OTHER ODD EXERCISE

1. For addition of ordered pairs, the sum of (x_1,y_1) and (x_2,y_2) is $(x_1 + x_2, y_1 + y_2)$. Thus, $(1,2) + (3,7) = (1 + 3, 2 + 7) = (4,9)$.

5. By addition of ordered pairs, $(1,2) + (0,y) = (1,2 + y)$. For this to be equal to $(1,3)$, we need $1 = 1$ and $2 + y = 3$. Thus, $y = 1$.

9. Using scalar multiplication, $2(1,b) + (b,4) = (2,2b) + (b,4)$. Then, addition of ordered pairs yields $(2 + b,2b + 4)$. Therefore, we need $2 + b = 3$ and $2b + 4 = 4$. For the first case, $b = 1$; and $b = 0$ for the second. Since b cannot be both 1 and 0 simultaneously, there is no solution.

13. Using scalar multiplication, $a(1,1) + b(1,-1)$ becomes $(a,a) + (b,-b)$. Then addition yields $(a + b,a - b) = (3,5)$. Solving $a + b = 3$ and $a - b = 5$ simultaneously, we get $a = 4$ and $b = -1$.

17. By addition of ordered pairs, $A + O = (x_1,y_1) + (0,0) = (x_1,y_1)$, which again is A .

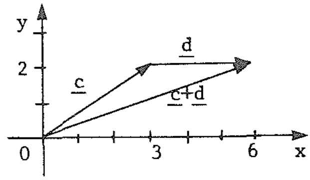
21. By scalar multiplication, the left-hand side is $a(bA) = a(b(x_1,y_1)) = a(bx_1,by_1) = (abx_1,aby_1)$. Similarly, $(ab)A = (ab)(x_1,y_1) = (abx_1,aby_1)$.

25. (a) We represent the molecule S_xO_y (x atoms of sulfur and y atoms of oxygen) by the ordered pair (x,y) . Then the chemical equation is equivalent to $k(1,3) + \ell(2,0) = m(1,2)$.

(b) From part (a), we have $k + 2\ell = m$ and $3k + 0 = 2m$.

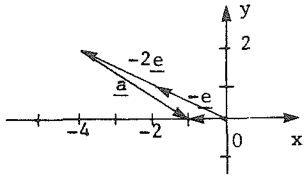
(c) From the second equation, we have $m = (3/2)k$. In order to make k and m integers, let $k = 2a$, so $m = 3a$. Substituting these values of k and m into the first equation gives $2a + 2\ell = 3a$, i.e., $\ell = a/2$. The smallest integer a , which would make ℓ an integer is $a = 2$. Thus, $k = 4$, $m = 6$, and $\ell = 1$.

29. (a)



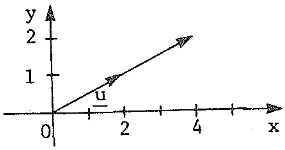
Using component notation, we have $\underline{c} + \underline{d} = (3,2) + (3,0) = (6,2)$. The vector $\underline{c} + \underline{d}$ is drawn analogous to Fig. 13.1.5.

(b)



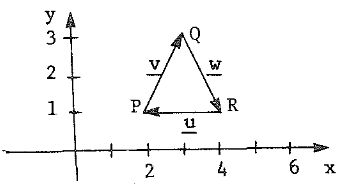
Component notation yields $-2\underline{e} + \underline{a} = -2(2,-1) + (3,-2) = (-4,2) + (3,-2) = (-1,0)$. $-2\underline{e}$ is constructed similar to the method in Fig. 13.1.8.

33.



First, draw $\underline{u}$. Then, extend the vector twice as far in the same direction to get $2\underline{u}$. The vector $2\underline{u}$ has components $(4,2)$.

37. (a)



(b) $\underline{v} = Q - P = (3,3) - (2,1) = (1,2)$; $\underline{w} = R - Q = (4,1) - (3,3) = (1,-2)$; $\underline{u} = P - R = (2,1) - (4,1) = (-2,0)$.

(c) $\underline{v} + \underline{w} + \underline{u} = (1,2) + (1,-2) + (-2,0) = (0,0) = \underline{0}$, the zero vector.

41. (a) Let $\underline{v} = (0,0)$, $\underline{w} = (1,1)$, and $s = 0$, then $r(0,0) + 0(1,1) = \underline{0}$ for any non-zero r . Thus, the definition of linear dependence is satisfied by $(0,0)$ and $(1,1)$.

(b) Intuitively, two vectors are parallel if and only if they have the same direction. Thus, if $\underline{v}$ and $\underline{w}$ are parallel, we can write $\underline{v} = -(s/r)\underline{w}$. This only makes sense if $r \neq 0$, so we can rearrange the equation to get $r\underline{v} + s\underline{w} = \underline{0}$. Therefore, parallel vectors are linearly dependent. Now, if the vectors are linearly dependent, we can go backwards from $r\underline{v} + s\underline{w} = \underline{0}$ to get $\underline{v} = -(s/r)\underline{w}$. Thus, linearly dependent vectors are parallel.

41. (c) If $a = 0$, then $c = 0$ and $ad = bc = 0$. If $b = 0$, then $d = 0$ and $ad = bc = 0$. If $a \neq 0$ and $b \neq 0$, then $r\underline{v} + s\underline{w} = 0$ if and only if $(ra,rb) + (sc,sd) = \underline{0}$, if and only if $ra + sc = 0$ and $rb + sd = 0$. Solve each of these equations for r and equate the results to get $-sc/a = -sd/b$. Cancel $-s$ and cross-multiply to get $ad = bc$ if and only if $r\underline{v} + s\underline{w} = \underline{0}$.

(d) Let $\underline{u} = (a,b)$, $\underline{v} = (c,d)$, and $\underline{w} = (e,f)$. Since $\underline{v}$ and $\underline{w}$ are linearly independent, $cf \neq de$ from part (c). Hence $c \neq 0$ or $d \neq 0$. Then $x\underline{v} + y\underline{w} = \underline{u}$ if and only if $(cx,dx) + (ey,fy) = (a,b)$, if and only if $cx + ey = a$ and $dx + fy = b$, if and only if $x = (a - ey)/c$ and $x = (b - fy)/d$ (and $c \neq 0$, $d \neq 0$). Hence, $(a - ey)d = (b - fy)c$ if and only if $ad - dey = bc - cfy$ if and only if $ad - bc = (de - cf)y$ if and only if $y = (ad - bc)/(de - cf)$. Substitute this for y in the equation for x: $x = [a - e(ad - bc)/(de - cf)]/c = [a(de - cf) - e(ad - bc)]/(c(de - cf)) = (ade - acf - ead + bce)/(c(de - cf)) = (be - af)/(de - cf)$. If $c = 0$ and $d \neq 0$, then $y = a/e = (ad - bc)/(de - cf)$ and $x = b - f(a/e)/d = (be - af)/(de - cf)$. Similarly, if $d = 0$ and $c \neq 0$.

Hence, for any $\underline{u}$, $\underline{v}$, and $\underline{w}$, we can specify x and y in terms of the coordinates of $\underline{u}$, $\underline{v}$, and $\underline{w}$.

SECTION QUIZ

1. On the xy-plane, draw a vector $\overrightarrow{PQ}$ and another vector $\overrightarrow{PR}$.

   (a) What is $\overrightarrow{PQ} - \overrightarrow{PR}$?

   (b) Sketch $\overrightarrow{PR} + \overrightarrow{PQ}$.

2. If the following vector expression is defined, simplify it. If not, explain why it is not defined: $3[(1,2+3)+4(2,5)-(-(2-1),-3)]$.

3. What is the difference between a vector and a scalar?

4. A chihuahua was minding his own business at $(1,0)$ when the mailman came to make his delivery. The little dog playfully followed the mailman down the street to $(0,0)$ and then around the corner to $(0,1)$. At that point, the irritable postman said, "Go away, dog! You bother me!" and kicked the chihuahua. In retaliation, the chihuahua bit the postman's leg.

   (a) What vector described the displacement of the dog along the first block?

   (b) What vector described the dog's total displacement?

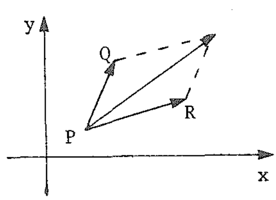
   Simultaneously, across town, a great dane followed a fearful mailman from $(98,25)$ to $(98,27)$, and then across a vacant lot to $(97,26)$.

   (c) Does the same vector describe the chihuahua's and the great dane's displacement? Explain your answer.*

ANSWERS TO SECTION QUIZ

1. (a) $\overrightarrow{RQ}$

   (b)



*Dear Reader: I realize that many of you hate math but are forced to complete this course for graduation. Thus, I have attempted to maintain interest with "entertaining" word problems. They are not meant to be insulting to your intelligence. Obviously, most of the situations will never happen; however, calculus has several practical uses and such examples are found throughout Marsden and Weinstein's text. I would appreciate your comments on whether my "unusual" word problems should be kept for the next edition.

2. $(30,84)$

3. A vector has magnitude and direction; a scalar has magnitude only.

4. (a) $(-1,0)$

   (b) $(-1,1)$

   (c) Yes; both are $(-1,1)$.

13.2 Vectors in Space

PREREQUISITES

1. Recall how to do algebra with ordered pairs (Section 13.1).
2. Recall how to do algebra with vectors in the plane (Section 13.1).

PREREQUISITE QUIZ

1. Simplify the following expressions:

   (a) $(3,5) + (-1,2)$

   (b) $-(1/2)(2,4)$

2. Let $P = (1,1)$, $Q = (-1,0)$, and $R = (2,-1)$. What is $3\overrightarrow{PQ} - \overrightarrow{QR}$?

GOALS

1. Be able to extend the ideas of the last section from ordered pairs to ordered triples.
2. Be able to apply the concept of vectors to problem solving.

STUDY HINTS

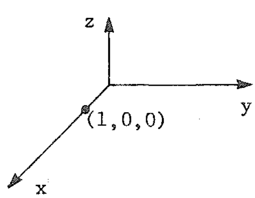
1. Right-hand rule. This will be important in Section 13.5 and also in physics courses. To determine the orientation, let the thumb of your right hand point in the positive z-direction. If your fingers, starting at the knuckles, curl from the positive x-axis to the positive y-axis, the orientation is right-handed. If not, it is left-handed. The right-handed orientation is the standard orientation.
2. Ordered triples. The algebra is analogous to that of ordered pairs. The notation used to denote the vectors or ordered triples is the same as for ordered pairs.

3. Notation. $\underline{i}$, $\underline{j}$, and $\underline{k}$ are the vectors with components (1,0,0) , (0,1,0) , and (0,0,1) . They point along the x , y , and z-axes, respectively. These are called the standard basis vectors. Similarly, $\underline{i}$ and $\underline{j}$ are (1,0) and (0,1) in the plane. Another special vector is $\underline{0}$, which is (0,0,0) , the zero vector.

4. Applications. Up until now, we have been very loose with the usage of the terms speed and velocity. From now on, you should know that speed is a scalar and velocity is a vector. Also, distance is a scalar and displacement is a vector.

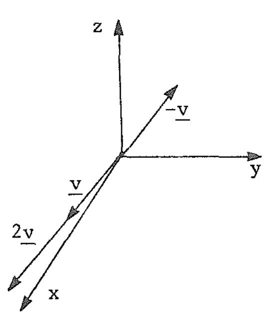
5. Problem solving. Since vectors have magnitude and direction, they can be represented pictorially. Thus, you should normally sketch a diagram to help you visualize a vector word problem.

SOLUTIONS TO EVERY OTHER ODD EXERCISE

1. 

The point (1,0,0) is located on the x-axis.

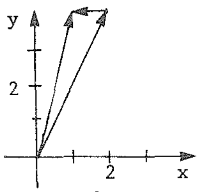
5. The sum of the ordered triples (x_1,y_1,z_1) and (x_2,y_2,z_2) is $(x_1 + x_2,y_1 + y_2,z_1 + z_2)$. Thus, (6,0,5) + (5,0,6) = (11,0,11) .

9. 

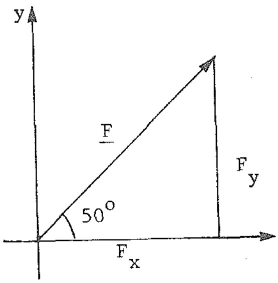
If $\underline{v}$ = (1,-1,-1) , then $2\underline{v}$ = (2,-2,-2) and $-\underline{v}$ = (-1,1,1) .

13. The vector with components (a,b,c) can be written as $a\underline{i} + b\underline{j} + c\underline{k}$. Thus, the given vector is $-\underline{i} + 2\underline{j} + 3\underline{k}$.

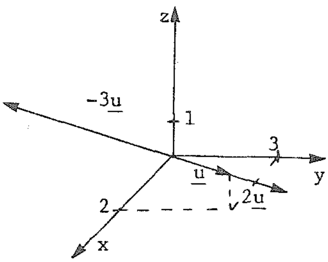
17. The vector from $(0,1,2)$ to $(1,1,1)$ is $(1,1,1) - (0,1,2) = (1,0,-1)$, i.e., $\underline{i} - \underline{k}$.

21. 

The vector joining the ship to the rock is $(2,4) - (1,0) = (1,4)$, i.e., $\underline{i} + 4\underline{j}$. As shown in the figure, $\tan\theta = 1/4$. Thus, the bearing of the rock from the ship is $\theta = \tan^{-1}(1/4) \approx 0.24$.

25. 

Using trigonometry, the horizontal component is $F_x = 50\cos(50^\circ) \approx 32.1$ lb , and the vertical component is $F_y = 50\sin(50^\circ) \approx 38.3$ lb .

29. 

All three vectors lie on the same line passing through the origin.

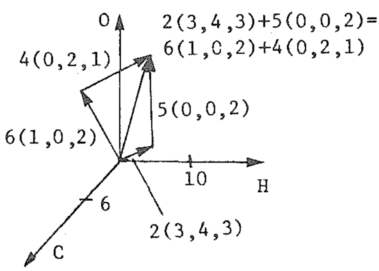
33. The components of $\underline{v}$ are $(3,4,5)$ and the components of $\underline{w}$ are $(1,-1,1)$. Using scalar multiplication and addition of ordered triples, we get $6\underline{v} + 8\underline{w} = 6(3,4,5) + 8(1,-1,1) = (18,24,30) + (8,-8,8) = (26,16,38)$, which corresponds to the vector $26\underline{i} + 16\underline{j} + 38\underline{k}$.

37. $a\underline{v} + b\underline{w} = a(\underline{i} + \underline{j}) + b(-\underline{i} + \underline{j}) = (a - b)\underline{i} + (a + b)\underline{j}$. Therefore, we need to simultaneously solve $a - b = 3$ and $a + b = 7$. The solution is $a = 5$ and $b = 2$.

41. (a) Let x be the number of atoms of C , y be the number of atoms of H , and z be the number of atoms of O . Mathematically, the chemical equation is $p(3,4,3) + q(0,0,2) = r(1,0,2) + s(0,2,1)$.

(b) We need to solve $3p = r$, $4p = 2s$, and $3p + 2q = 2r + s$. Suppose $p = 1$, then $r = 3$, $s = 2$, and $q = 5/2$. The smallest integers are obtained by multiplying by 2 , i.e., $p = 2$, $r = 6$, $s = 4$, and $q = 5$.

(c)



SECTION QUIZ

1. Which of the following notations can be used to represent a vector in space?

(a) $-\underline{v}$

(b) $\overrightarrow{RL}$

(c) $\underline{n} + \underline{o}$

(d) $(2,0,1)$

(e) $(\underline{x},\underline{y},\underline{z})$

2. If a vector $\underline{v}$ originates at $(1,-2,1)$ and extends to $(0,3,2)$, write $-2\underline{v}$ with the standard basis notation.

3. (a) Is north a vector? Why or why not?

   (b) Is velocity a vector? Why or why not?

4. A motorboat in the middle of a bay is travelling in a northwesterly direction at a speed of 4 knots. A wind is blowing due south at 2 knots, and the current is flowing due east at 1 knot. What would be the motorboat's speed and direction in the absence of both the wind and the current?

5. An old prospector had lost his compass and was lost in the middle of the desert when he came upon a rattlesnake. Keeping an eye on the rattler, the old prospector slowly backed into a cactus. The pain caused him to shoot into the air with velocity $2\underline{i} + 3\underline{j} + \underline{k}$. Gravity is causing him to descend with velocity $-(1/2)\underline{k}$ and a wind is blowing him away with velocity $2\underline{i} - \underline{j}$. What is the net velocity vector of the poor prospector?

ANSWERS TO PREREQUISITE QUIZ

1. (a) (2,7)

   (b) (-1,-2)

2. (-9,-2)

ANSWERS TO SECTION QUIZ

1. a,b,c,d

2. $-2\underline{i} + 10\underline{j} - 2\underline{k}$

3. (a) No; it is only a direction; it has no magnitude.

   (b) Yes; it has both magnitude and direction.

4. Speed = $[21 + 12\sqrt{2}]^{1/2}$ knot ; direction = 0.66 radians north of west.

5. $4\underline{i} + 2\underline{j} + (1/2)\underline{k}$

13.3 Lines and Distance

PREREQUISITES

1. Recall how to perform vector algebra in space (Section 13.2).
2. Recall the triangle inequality (Section R.2).
3. Recall how to find the equations of lines and the distance between two points in the plane (Section R.4).

PREREQUISITE QUIZ

1. Let $\underline{u} = \underline{i} - \underline{j}$, $\underline{v} = 2\underline{j} + \underline{k}$, and $\underline{w} = \underline{i} + \underline{j} + \underline{k}$. What is $3\underline{u} - \underline{v} + 2\underline{w}$?
2. Let $(-1,-1)$, $(1,1)$, and $(0,1)$ be the vertices of a triangle.
   (a) Find the length of the three sides.
   (b) Show how the triangle inequality applies to this triangle.
3. What is the equation of the line passing through the points $(2,2)$ and $(3,1)$?

GOALS

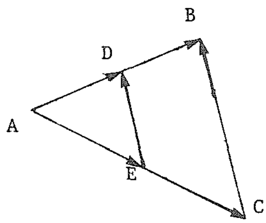
1. Be able to write the equation of a line in the plane or in space given two points or a point and a direction.
2. Be able to normalize a given vector.

STUDY HINTS

1. Point-point form. You should memorize the formula for a line: $R = (1 - t)P + tQ$, where $R = (x,y,z)$, $P = (x_1,y_1,z_1)$, $Q = (x_2,y_2,z_2)$. To see if the direction is correct, plug in $t = 0$ — you should get the first point. Substituting $t = 1$ should give the second point. Look at Example 4 to see how the two point-point forms are equivalent.

2. Point-direction form. You should memorize $R = P + t\underline{d}$, where R , P , and $\underline{d}$ are defined according to the box on p. 663. From this, it is simple to get the component form. If you must find the direction vector $\underline{d}$ from two given points, substituting $t = 0$ and $t = 1$ to recover the given points will tell you if your direction $\underline{d}$ is correct.
3. Lines in the plane. The same equations hold if the lines lie in the xy-plane. Simply let $z = 0$.
4. Non-intersecting lines in space. Unlike lines in the plane, two lines in space will not generally intersect. See Example 6(a). As a side note, two non-intersecting lines in space are called skew.
5. Length of a vector. It is denoted by $\|\underline{v}\|$ and it is equal to $\sqrt{a^2 + b^2 + c^2}$ if $\underline{v} = a\underline{i} + b\underline{j} + c\underline{k}$. Note how this is similar to the length of the line segment drawn in the plane from $(0,0)$ to (a,b) , which is $\sqrt{a^2 + b^2}$.
6. Properties of length. Understand why the properties on p. 665 are reasonable to expect; with this understanding, you need not memorize them.
7. Normalization. If $\|\underline{u}\| = 1$, then $\underline{u}$ is called a unit vector. Dividing a vector $\underline{v}$ by its length $\|\underline{v}\|$ gives a unit vector, provided $\|\underline{v}\| \neq 0$. The process of creating a unit vector in this fashion is called normalization of $\underline{v}$.
8. Distance formula. As with the formula for length, we just insert an extra term to the planar formula; $+ (z_2 - z_1)^2$ in this case. The squaring process permits one to reverse the subscripts without changing the answer.

SOLUTIONS TO EVERY OTHER ODD EXERCISE

1. 

Let $\underline{v}_1$ be the vector joining A to B, and let $\underline{v}_2$ be the vector joining A to C. Then, the vector joining A to D is $(1/2)\underline{v}_1$ and the vector joining A to E is $(1/2)\underline{v}_2$. Thus, the vector joining E to D is $(1/2)\underline{v}_1 - (1/2)\underline{v}_2$, i.e., $(1/2)(\underline{v}_1 - \underline{v}_2)$. Since the vector joining C to B is $\underline{v}_1 - \underline{v}_2$, $\overline{DE}$ is parallel to $\overline{BC}$ and has one-half the length of $\overline{BC}$.

5. Let $\underline{v}_1$ be the vector from the origin to the point $(1,1)$ and $\underline{v}_2$ be the vector from the origin to the point $(2,-2)$. Then, from the results of Example 1, the vector from the origin to the midpoint of the line segment between $(1,1)$ and $(2,-2)$ is $(1/2)(\underline{v}_1 + \underline{v}_2) = (3/2,-1/2)$. Thus, the desired point is $(2/3)(3/2,-1/2) = (1,-1/3)$.

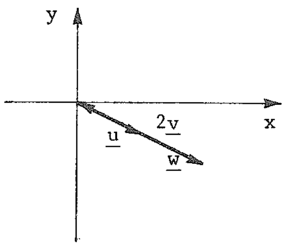
9. Using the method of Example 4, we obtain as the equation of the line $(x,y,z) = (1 - t)(0,0,0) + t(1,1,1) = (t,t,t)$. Alternatively, it can be expressed as $x = t$, $y = t$, and $z = t$.

13. This is analogous to Example 5(b). The line is described by $(x,y) = (-1,-2) + t(3,-2) = (-1 + 3t,-2 - 2t)$. Thus, $x = -1 + 3t$ and $y = -2 - 2t$.

17. If the lines intersect, then for some t_1 and t_2, we need $t_1 = 3t_2$, $3t_1 - 1 = 5$, and $4t_1 = 1 - t_2$. $3t_1 - 1 = 5$ implies $t_1 = 2$. Substituting $t_1 = 2$ into $t_1 = 3t_2$ yields $t_2 = 2/3$. However, substituting these values into the last equation gives $8 = 1/3$, which is false. Thus, the lines do not intersect.

21. The length of a vector is $\|\underline{v}\| = \|a\underline{i} + b\underline{j} + c\underline{k}\| = \sqrt{a^2 + b^2 + c^2}$. In this case, $\|\underline{v}\| = \sqrt{1 + 0 + 1} = \sqrt{2}$.

25. The length of the vector is $\|a\mathbf{i} - 3\mathbf{j} + \mathbf{k}\| = \sqrt{a^2 + 9 + 1} = \sqrt{10 + a^2} = 16$. Squaring yields $10 + a^2 = 256$, i.e., $a^2 = 246$, i.e., $a = \pm\sqrt{246}$.

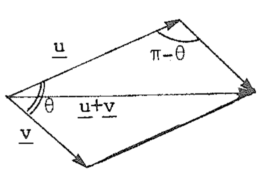
29. 

$\mathbf{u} + 2\mathbf{v} + \mathbf{w} = \mathbf{0}$ implies $\mathbf{u} + \mathbf{w} = -2\mathbf{v}$. Since $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are unit vectors, we can only have $\mathbf{u} = \mathbf{w} = -\mathbf{v}$. A possible solution is $\mathbf{u} = \mathbf{i}$, $\mathbf{v} = -\mathbf{i}$, and $\mathbf{w} = \mathbf{i}$. Another possibility is shown in the diagram.

33. The normalization of $\mathbf{v}$ is $\mathbf{v}/\|\mathbf{v}\|$. For $\mathbf{v} = \mathbf{i} + \mathbf{j} + \mathbf{k}$, $\|\mathbf{v}\| = \sqrt{1 + 1 + 1} = \sqrt{3}$; the normalization is $(1/\sqrt{3})\mathbf{i} + (1/\sqrt{3})\mathbf{j} + (1/\sqrt{3})\mathbf{k}$. For $\mathbf{v} = \mathbf{i} + \mathbf{k}$, $\|\mathbf{v}\| = \sqrt{1 + 0 + 1} = \sqrt{2}$; the normalization is $(1/\sqrt{2})\mathbf{i} + (1/\sqrt{2})\mathbf{k}$.

37. The distance from (x_1, y_1, z_1) to (x_2, y_2, z_2) is $[(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2]^{1/2}$. In this case, the distance is $[(1 - 1)^2 + (1 - 2)^2 + (2 - 3)^2]^{1/2} = \sqrt{2}$.

41. Let $\mathbf{v}_1$ be the velocity vector of the boat at full speed, i.e., $\mathbf{v}_1 = (0,12)$. Let $\mathbf{v}_2$ be the velocity vector of the current, i.e., $\mathbf{v}_2 = (5,0)$. Then the velocity vector of the boat in the current is $\mathbf{v}_1 + \mathbf{v}_2 = (5,12)$. Hence, the speed of the boat is $\|\mathbf{v}_1 + \mathbf{v}_2\| = \sqrt{5^2 + 12^2} = 13$ knots.

45. 

Using the law of cosines, $\|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - \|\mathbf{u}\|\|\mathbf{v}\| \cos(\pi - \theta) = (\|\mathbf{u}\| + \|\mathbf{v}\|)^2$ if $\cos(\pi - \theta) = -1$, i.e., $\pi - \theta = \pi$ or $\theta = 0$. Equality holds if the angle between the two vectors is 0 , i.e., $\mathbf{u} = a\mathbf{v}$ for any scalar a . Using the vectors in Exercises 21 and 23, we have $\|\mathbf{u} + \mathbf{v}\| = \|3\mathbf{i} + 3\mathbf{k}\| = 3\sqrt{2}$ while $\|\mathbf{u}\| + \|\mathbf{v}\| = \sqrt{2} + 2\sqrt{2} = 3\sqrt{2} = \|\mathbf{u} + \mathbf{v}\|$. Equality is expected since $2\mathbf{i} + 2\mathbf{k} = 2(\mathbf{i} + \mathbf{k})$.

SECTION QUIZ

1. Write the equations describing the following lines:

   (a) The line from $(1,3,2)$ to $(-1,-3,-3)$.

   (b) The line containing $(-1,2,1)$ with direction $2\underline{i} - 3\underline{j} + \underline{k}$.

2. True or false. The line $(1,0,0) + t(0,1,0)$ does not contain $(1,-1,0)$ because t would have to be negative.

3. For the line in Question 1(a), find a point on the line which is one unit distant from $(1,3,2)$.

4. Find unit vectors which have the following directions:

   (a) $2\underline{i} + 3\underline{j} + \underline{k}$

   (b) $3\underline{j} - \underline{i} + \underline{k}$

   (c) $4\underline{j}$

5. Bert the Beaver had unusually fast-growing teeth, so he had to gnaw more wood than most other beavers. This was ideal for Bert because he lived on an island in the middle of the river. Since Bert gnawed more wood, it was easy for him to build a bridge to his island, which is known as Bert's Hill. The latest tree which fell due to Bert's gnawing originated at $(2,3,1)$ and ended up at $(3,1,2)$.

   (a) If the bridge follows a straight line, what is its equation?

   (b) What unit vector describes the bridge's direction?

   (c) How long is the bridge?

ANSWERS TO PREREQUISITE QUIZ

1. $5\underline{i} - 3\underline{j} + \underline{k}$

2. (a) $2\sqrt{2}$, 1 , and $\sqrt{5}$

   (b) $1 + \sqrt{5} \geqslant 2\sqrt{2}$; $2\sqrt{2} + 1 \geqslant \sqrt{5}$; or $2\sqrt{2} + \sqrt{5} \geqslant 1$

3. $y = -x + 4$

ANSWERS TO SECTION QUIZ

1. (a) $(1,3,2) + (-2,-6,-5)t$

   (b) $(-1,2,1) + (2,-3,1)t$

2. False; t may have any real value.

3. $(1,3,2) \pm (2,6,5)/\sqrt{65}$

4. (a) $(2\underline{i} + 3\underline{j} + \underline{k})/\sqrt{14}$

   (b) $(-\underline{i} + 3\underline{j} + \underline{k})/\sqrt{11}$

   (c) $\underline{j}$

5. (a) $(2,3,1) + (1,-2,1)t$

   (b) $(\underline{i} - 2\underline{j} + \underline{k})/\sqrt{6}$

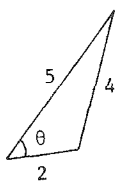
   (c) $\sqrt{6}$

13.4 The Dot Product

PREREQUISITES

1. Recall the algebraic and geometric properties of vectors (Sections 13.1 and 13.2).
2. Recall how to use the law of cosines (Section 5.1).
3. Recall how to use the inverse cosine function (Section 5.4).

PREREQUISITE QUIZ

1. What is the length of $\underline{v} = 2\underline{i} + \underline{j} - 2\underline{k}$?
2. 

   (a) Write the equation which describes the law of cosines.

   (b) Use the law of cosines to determine θ .
3. Compute $\cos^{-1}(\sqrt{3}/2)$.
4. Compute $\cos^{-1}(-1/2)$.

GOALS

1. Be able to state the definition of the dot product and compute it.
2. Be able to compute an orthogonal projection.
3. Be able to write the equation of a plane, given the appropriate data.
4. Be able to find the distance from a point to a line or a plane.

STUDY HINTS

1. Dot product. You should memorize the formulas $\underline{v}_1 \cdot \underline{v}_2 = a_1a_2 + b_1b_2 + c_1c_2 = \|\underline{v}_1\| \cdot \|\underline{v}_2\| \cos\theta$ where $\underline{v}_1 = a_1\underline{i} + b_1\underline{j} + c_1\underline{k}$, $\underline{v}_2 = a_2\underline{i} + b_2\underline{j} + c_2\underline{k}$, and θ is the angle between $\underline{v}_1$ and $\underline{v}_2$. The dot product is also known as the scalar product or the inner product. Note that the

1. (continued)

   dot product is always a real number. To get the dot product in the plane, simply let $c_1 = c_2 = 0$.

2. Geometry of dot product. Since $\underline{v}_1 \cdot \underline{v}_2 = \|\underline{v}_1\| \cdot \|\underline{v}_2\| \cos\theta$, it is now possible to determine the angle θ between two vectors. If $\theta = 0$, we get the statement that $\|\underline{v}\|^2 = \underline{v} \cdot \underline{v}$. If $\theta = \pi/2$, $\cos\theta = 0$; therefore, two vectors are perpendicular when their dot product is 0 . The important formulas derived in this section are based upon this last fact.

3. Dot product properties. Properties 1 and 2, listed at the bottom of the box on p. 669, are obvious. The next three properties are analogous to properties of multiplication from algebra. Since $|\cos\theta| \leqslant 1$, we get property 6 from the geometric interpretation. When $\cos\theta = 0$, θ is $\pi/2$, so the vectors are perpendicular.

4. Orthogonal projections. The orthogonal projection of $\underline{v}$ on $\underline{u}$ is depicted by placing $\underline{u}$ and $\underline{v}$ such that they originate from the same point. Then draw a perpendicular from the line defined by $\underline{u}$ and extend it to the tip of $\underline{v}$. See Fig. 13.4.3. The projection is simply the "shadow" of $\underline{v}$ along $\underline{u}$; it is always a multiple of $\underline{u}$. It is probably best to remember the formula for the orthogonal projection of $\underline{v}$ on $\underline{u}$: $[(\underline{v} \cdot \underline{u})/(\underline{u} \cdot \underline{u})]\underline{u}$. As usual, understanding the formula will help you to remember it .

5. Point to line distance. Refer to Fig. 13.4.4. We know the coordinates of P and Q , so it is simple to determine $\|\overrightarrow{PQ}\|$. $\overrightarrow{PR}$ is simply an orthogonal projection, so $\|\overrightarrow{PR}\|$ can be determined. Finally, using the Pythagorean theorem, one can determine $\|\overrightarrow{RQ}\|$. You should get formula (4).

6. Equation of a plane. Let (x,y,z) and (x_0,y_0,z_0) be two points in the plane, so the vector from one point to the other is $\underline{v} = (x - x_0)\underline{i} + (y - y_0)\underline{j} + (z - z_0)\underline{k}$. If $\underline{n} = A\underline{i} + B\underline{j} + C\underline{k}$ is perpendicular to the plane, it is also perpendicular to $\underline{v}$, so $\underline{n}\cdot\underline{v} = 0$, which gives the equation of the plane, formula (5). Formula (6) simply substitutes D for $-Ax_0 - By_0 - Cz_0$. It is best to memorize these formulas.

7. Plane through 3 points. Three points not lying on a line determine a plane. To obtain the equation of the plane, remember that all three points satisfy $Ax + By + Cz + D = 0$. Solve the resulting three equations for A , B , and C in terms of D . Then choose D arbitrarily. (See Example 8.)

8. Finding a plane's equation. In general, three equations are needed to determine the numbers A , B , and C in the equation of the plane.

9. Point to plane distance. Equation (7) is derived in a fashion analogous to the derivation of Equation (4). $\underline{n}$ is chosen to be a unit vector in order to simplify things. This made $\underline{n}\cdot\underline{n} = 1$ and when we computed the length of the projection, we used $\|\underline{n}\| = 1$. Formula (7) is something you may wish to memorize rather than derive.

SOLUTIONS TO EVERY OTHER ODD EXERCISE

1. The dot product of $a_1\underline{i} + b_1\underline{j} + c_1\underline{k}$ and $a_2\underline{i} + b_2\underline{j} + c_2\underline{k}$ is $a_1a_2 + b_1b_2 + c_1c_2$. In this case, $(\underline{i} + \underline{j} + \underline{k})\cdot(\underline{i} + \underline{j} + 2\underline{k}) = 1\cdot 1 + 1\cdot 1 + 1\cdot 2 = 4$.

5. Use the formula $\theta = \cos^{-1}(\underline{v}_1\cdot\underline{v}_2/\|\underline{v}_1\|\|\underline{v}_2\|)$. Let $\underline{v}_1 = \underline{i} + \underline{j} + \underline{k}$, so $\|\underline{v}_1\| = \sqrt{3}$. Let $\underline{v}_2 = \underline{i} + \underline{j} + 2\underline{k}$, so $\|\underline{v}_2\| = \sqrt{6}$. Thus, $\theta = \cos^{-1}(4/\sqrt{3}\sqrt{6}) = \cos^{-1}(4/3\sqrt{2}) \approx 0.34$ radian.

9. We want $\underline{v} = a\underline{i} + b\underline{j}$ such that $2a - b = 0$ and $\sqrt{a^2 + b^2} = 1$. A simple method is to find an orthogonal vector and then normalize it. Such a vector is $\underline{i} + 2\underline{j}$, so $\underline{v} = (1/\sqrt{5})\underline{i} + (2/\sqrt{5})\underline{j}$.

13. From Example 4, the orthogonal projection of $\underline{v}$ on $\underline{u}$ is the vector $(\underline{v}\cdot\underline{u}/\underline{u}\cdot\underline{u})\underline{u}$. Thus, the length of this vector is $\|(\underline{v}\cdot\underline{u}/\underline{u}\cdot\underline{u})\underline{u}\| = |\underline{v}\cdot\underline{u}/\underline{u}\cdot\underline{u}|\|\underline{u}\|$. Since $\underline{u}\cdot\underline{u} = \|\underline{u}\|^2$, the length is $|\underline{v}\cdot\underline{u}|/\|\underline{u}\| = (|\underline{v}\cdot\underline{u}|/\|\underline{u}\|\|\underline{v}\|)\|\underline{v}\| = |\cos\theta|\|\underline{v}\|$, where θ is the angle between $\underline{u}$ and $\underline{v}$.

17. Use the formula $\mathrm{dist}(Q,\ell) = \{(x_1 - x_0)^2 + (y_1 - y_0)^2 + (z_1 - z_0)^2 - [a(x_1 - x_0) + b(y_1 - y_0) + c(z_1 - z_0)]^2/(a^2 + b^2 + c^2)\}^{1/2}$. The line $x = 3t + 2$, $y = -t - 1$, $z = t + 1$ goes through $(2,-1,1) = (x_0,y_0,z_0)$ and has direction $(3,-1,1) = (a,b,c)$. With $(x_1,y_1,z_1) = (1,1,2)$, the distance is $\{(-1)^2 + (-2)^2 + (1)^2 - [3(-1) + (-1)(-2) + (1)(1)]^2/(9 + 1 + 1)\}^{1/2} = \sqrt{6 - 0/11} = \sqrt{6}$.

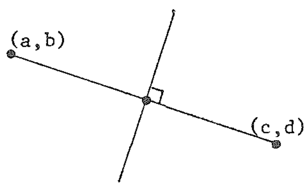
21. The equation of the plane through (x_0,y_0,z_0) with normal vector $A\underline{i} + B\underline{j} + C\underline{z}$ is $A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$. Thus, the plane is $1(x - 0) + 0(y - 0) + 0(z - 0) = 0$, i.e., $x = 0$.

25. From the coefficients, a normal vector is $2\underline{i} + 3\underline{j} + \underline{k}$. Its length is $\sqrt{4 + 9 + 1} = \sqrt{14}$, so a normal unit vector is $(2/\sqrt{14})\underline{i} + (3/\sqrt{14})\underline{j} + (1/\sqrt{14})\underline{k}$.

29. Since the general equation of the plane is $Ax + By + Cz + D = 0$, we can substitute the given points to get $B + D = 0$, $A + D = 0$, and $C + D = 0$. Therefore $D = -A = -B = -C$. Arbitrarily choosing $D = -1$ yields $A = B = C = 1$, and the equation of the plane becomes $x + y + z - 1 = 0$.

33. The distance from (x_1,y_1,z_1) to the plane $Ax + By + Cz + D = 0$ is $|Ax_1 + By_1 + Cz_1 + D|/\sqrt{A^2 + B^2 + C^2}$. The line is parametrically represented by $x = 2t$, $y = -t$, and $z = 3t$. It meets the plane when $2x - y + 3z = 4t + t + 9t = 7$, i.e., $14t = 7$, i.e., $t = 1/2$. The point of intersection is $(1,-1/2,3/2)$. The distance from $(0,0,0)$ to the plane is $|(2)(0) - (1)(0) + (3)(0) - 7|/\sqrt{4 + 1 + 9} = 7/\sqrt{14} = \sqrt{14}/2$.

37. Since the distance from (x_1,y_1,z_1) to the plane $Ax + By + Cz + D = 0$ is $|Ax_1 + By_1 + Cz_1 + D|/\sqrt{A^2 + B^2 + C^2}$, we have, in this particular case, distance $= |(1)(1) - (1)(1) - (1)(1) + 10|/\sqrt{1 + 1 + 1} = 9/\sqrt{3} = 3\sqrt{3}$.

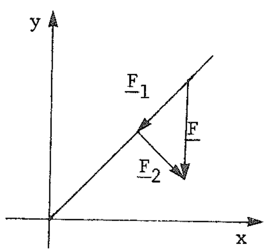
41. 

Let the coordinates of the two points be (a,b) and (c,d) . The line that we are looking for contains the point $((a + c)/2, (b + d)/2)$ and has normal vector $(a - c)\underline{i} + (b - d)\underline{j}$. Thus, the equation of the line has the form $(a - c)x + (b - d)y = E$. Since the midpoint $((a + c)/2,(b + d)/2)$ is on the line, we have $(a - c)((a + c)/2) + (b - d)((b + d)/2) = E$. So the equation of the line is $(a - c)x + (b - d)y = (1/2)(a^2 - c^2 + b^2 - d^2)$. We need to show that any (x,y) satisfying the above equation is equidistant from the two points (a,b) and (c,d) . We have $y = -[(a - c)/(b - d)]x + [(a^2 - c^2 + b^2 - d^2)/2(b - d)]$, so $(x - a)^2 + (y - b)^2 = (x - a)^2 + \{-[(a - c)/(b - d)]x + (a^2 - c^2 + b^2 - d^2)/2(b - d) - b\}^2 = (x - a)^2 + [((a^2 - 2ax + x^2) - (c^2 - 2cx + x^2) - (b^2 - 2bd + d^2))/2(b - d)]^2 = (x - a)^2 + [((a - x)^2 - (c - x)^2 - (b - d)^2)/2(b - d)]^2$. On the other hand, $(x - c)^2 + (y - d)^2 = (x - c)^2 + [((a - x)^2 - (c - x)^2 + (b - d)^2)/2(b - d)]^2$. Putting the

41. (continued)

last two expressions over a common denominator $4(b - d)^2$, we see that they are both equal to $[(a - x)^4 + (c - x)^4 + (b - d)^4 - 2(a - x)^2(c - x)^2 + 2(a - x)^2(b - d)^2 + 2(c - x)^2(b - d)^2]/4(b - d)^2$. Thus, (x,y) on the line is indeed equidistant from (a,b) and (c,d).

45. Let $P_1 = (p,q)$ and $P_2 = (r,s)$. We need $(x - p)^2 + (y - q)^2 = (x - r)^2 + (y - s)^2$, so $(x - p)^2 - (x - r)^2 = (y - s)^2 - (y - q)^2$. Factoring gives $(x - p + x - r)(x - p - x + r) = (y - s + y - q) \times (y - s - y + q)$, so $(2x - p - r)(r - p) = (2y - s - q)(q - s)$, so $(2x)(r - p) + p^2 - r^2 = 2y(q - s) + s^2 - q^2$, so $2x(r - p) + 2y(s - q) = r^2 + s^2 - p^2 - q^2$. Hence, $ax + by = c$ if $a = r - p$, $b = s - q$, and $c = (r^2 + s^2 - p^2 - q^2)/2$.

49. 

A unit vector in the direction of the plane is $\underline{e}_1 = (1/\sqrt{2})(-\underline{i} - \underline{j})$, and one perpendicular to it is $\underline{e}_2 = (1/\sqrt{2})(\underline{i} - \underline{j})$. Write $\underline{F} = -F\underline{j}$, where $F = \|\underline{F}\|$ is the magnitude of $\underline{F}$. Then we write $\underline{F} = \alpha\underline{e}_1 + \beta\underline{e}_2 = (\alpha/\sqrt{2})(-\underline{i} - \underline{j}) + (\beta/\sqrt{2})(\underline{i} - \underline{j}) = -F\underline{j}$. This will hold if $-\alpha + \beta = 0$ and $-\alpha/\sqrt{2} - \beta/\sqrt{2} = -1$, i.e., if $\alpha = \beta$ and $\alpha + \beta = \sqrt{2}$, i.e., $\alpha = \beta = 1/\sqrt{2}$. Thus, $\underline{F} = \underline{F}_1 + \underline{F}_2$, where $\underline{F}_1 = -(F/2)(\underline{i} + \underline{j})$ and $\underline{F}_2 = (F/2)(\underline{i} - \underline{j})$.

53. Let $\underline{u} = x\underline{i} + y\underline{j} + z\underline{k}$.

(a) $\underline{u}\cdot\underline{u} = x^2 + y^2 + z^2 \geqslant 0$.

(b) If $\underline{u}\cdot\underline{u} = 0$, then $x^2 + y^2 + z^2 = 0$, so $x = y = z = 0$, and $\underline{u} = \underline{0}$.

(c) Let $\underline{v} = r\underline{i} + s\underline{j} + t\underline{k}$. Then $\underline{u}\cdot\underline{v} = xr + ys + zt$ and $\underline{v}\cdot\underline{u} = rx + sy + tz = \underline{u}\cdot\underline{v}$.

53. (d) Let $\underline{w} = \ell\underline{i} + m\underline{j} + n\underline{k}$. Then $(a\underline{u} + b\underline{v})\cdot\underline{w} = [(ax + br)\underline{i} + (ay + bs)\underline{j} + (az + bt)\underline{k}]\cdot(\ell\underline{i} + m\underline{j} + n\underline{k}) = (a\ell x + b\ell r)\underline{i} + (amy + bms)\underline{j} + (anz + bnt)\underline{k}$. Also $a(\underline{u}\cdot\underline{w}) + b(\underline{v}\cdot\underline{w}) = a(x\ell + ym + zn) + b(r\ell + sm + tn) = (a\ell x + b\ell r)\underline{i} + (amy + bms)\underline{j} + (anz + bnt)\underline{k} = (a\underline{u} + b\underline{v})\cdot\underline{w}$.

57. (a) Equating components, we get $\mu = a/\sqrt{a^2 + b^2 + c^2}$, $\lambda = b/\sqrt{a^2 + b^2 + c^2}$, and $\nu = c/\sqrt{a^2 + b^2 + c^2}$. Hence, if $s = t\sqrt{a^2 + b^2 + c^2}$, then $P_0 + s(\mu,\lambda,\nu) = P_0 + t\sqrt{a^2 + b^2 + c^2} \times (a/\sqrt{a^2 + b^2 + c^2}, b/\sqrt{a^2 + b^2 + c^2}, c/\sqrt{a^2 + b^2 + c^2}) = P_0 + t(a,b,c)$, which is the same line.

(b) Note that $\|\underline{u}\| = \|\underline{d}\|/\|\underline{d}\| = 1$. Hence, $\underline{i}\cdot\underline{u} = \mu = \|\underline{i}\|\|\underline{u}\| \cos\alpha = \cos\alpha$; $\underline{j}\cdot\underline{u} = \lambda = \|\underline{j}\|\|\underline{u}\| \cos\beta = \cos\beta$; and $\underline{k}\cdot\underline{u} = \underline{\nu} = \|\underline{k}\|\|\underline{u}\| \cos\gamma = \cos\gamma$.

(c) $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = \mu^2 + \lambda^2 + \nu^2 = \underline{u}\cdot\underline{u} = \|\underline{u}\|^2 = 1$.

(d) For L_1 , $\cos\alpha = 1/\sqrt{3}$ so $\alpha = \cos^{-1}(1/\sqrt{3})$; $\cos\beta = 1/\sqrt{3}$, so $\beta = \cos^{-1}(1/\sqrt{3})$; and $\cos\gamma = 1/\sqrt{3}$, so $\gamma = \cos^{-1}(1/\sqrt{3})$. L_2 has the same direction angles and cosines as L_1 , since its direction is the same. For L_3 , $\cos\alpha = 1/\sqrt{83}$, so $\alpha = \cos^{-1}(1/\sqrt{83})$; $\cos\beta = 1/\sqrt{83}$, so $\beta = \cos^{-1}(1/\sqrt{83})$; and $\cos\gamma = 9/\sqrt{83}$, so $\gamma = \cos^{-1}(9/\sqrt{83})$. L_4 has the same direction angles and cosines as L_3 , since it has the same direction.

(e) $\alpha = \beta = \gamma$ implies $\cos\alpha = \cos\beta = \cos\gamma$. Thus, $\mu = \lambda = \nu$, so $a = b = c$. Hence only the line $t(1,1,1)$ has direction angles $\alpha = \beta = \gamma$.

SECTION QUIZ

1. Compute the following dot products:

   (a) $(\underline{i} + \underline{j} - 3\underline{k})\cdot(-2\underline{j} - \underline{k})$

   (b) $(2\underline{i} + 3\underline{j})\cdot(\underline{i})$

2. Let ℓ be the line $(1,5,-2) + t(-2,3,1)$. What is an equation of a line perpendicular to ℓ and which passes through $(5,-1,6)$?

3. A plane P contains the points $(1,1,1)$, $(2,3,0)$, and $(2,5,-1)$. Find the equation of a parallel plane containing the origin.

4. In the plane, find y if the vector $2\underline{i} + y\underline{j}$ is $\pi/3$ radians from the vector $-3\underline{i} + 4\underline{j}$.

5. Let $\underline{u} = (1,2,3)$ and $\underline{v} = (4,3,2)$. What is the orthogonal projection of $\underline{v}$ on $\underline{u}$ and the orthogonal projection of $\underline{u}$ on $\underline{v}$?

6. A proud unicorn has its horn extending from the point $P(2,1,1)$ to the point $Q(3,2,3)$. A magic mirror containing the image of an evil wizard exists on the plane $2x - 3y + z = 0$.

   (a) Suppose the evil wizard's ugly face provokes the unicorn into attacking. How far does the unicorn need to move, i.e., what is the distance from Q to the plane?

   (b) The sun is situated so that a shadow is cast on the ground in the direction $-\underline{i} - \underline{j}$. How long is the shadow, i.e., what is the length of the projection of the horn's vector upon $-\underline{i} - \underline{j}$?

ANSWERS TO PREREQUISITE QUIZ

1. 3

2. (a) $a^2 + b^2 - 2ab\cos\theta = c^2$

   (b) $\theta = \cos^{-1}(13/20) \approx 0.86$

3. $\pi/6$

4. $2\pi/3$

ANSWERS TO SECTION QUIZ

1. (a) 1

   (b) 2

2. $(5,-1,6) + (6,-9,39)t$ is one possible answer.

3. $y + 2z = 0$

4. $y = (96 \pm 50\sqrt{3})/39$

5. $(16/\sqrt{14})(1,2,3)$ and $(16/\sqrt{29})(4,3,2)$

6. (a) $3/\sqrt{14}$

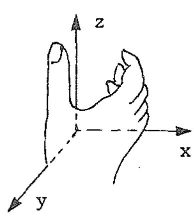
   (b) $\sqrt{2}$

13.5 The Cross Product

PREREQUISITES

1. Recall how to compute the equation of a plane, given three points in the plane (Section 13.4).
2. Recall the right-hand rule (Section 13.2).

PREREQUISITE QUIZ

1. Find the equation of the plane passing through the points $(1,0,0)$, $(0,3,0)$, and $(-1,2,3)$.
2. 

In the diagram, the right hand is the shown. Do the x-, y-, and z-axes form a right-handed or a left-handed system?

GOALS

1. Be able to state the definition of the cross product and compute it.
2. Be able to use the cross product to find the area of a parallelogram and to find the equation of a plane.

STUDY HINTS

1. Cross products. The cross product, also known as the vector product, is always a vector, unlike the dot product, which is a scalar. Do not memorize the component formula. You will be given an easier formula in the next section. For now, keep referring to formula (1) or use the distributive property of cross products along with Fig. 13.5.5. Note the clockwise positioning of $\underset{\sim}{i}$, $\underset{\sim}{j}$, and $\underset{\sim}{k}$ in the figure.

2. Cross product geometry. The cross product $\underline{u} \times \underline{v}$ is perpendicular to both $\underline{u}$ and $\underline{v}$. $\underline{u}$, $\underline{v}$, and $\underline{u} \times \underline{v}$, in this order, form a right-handed system. Also, the length of the cross product is the area of the parallelogram spanned by $\underline{u}$ and $\underline{v}$. Note that the cross product is related to $\sin\theta$, whereas the dot product is related to $\cos\theta$, where θ is the angle between the two vectors.

3. Algebraic properties. If the cross product is zero, then either: (i) the length of one of the vectors must be zero, or (ii) $\sin\theta = 0$, i.e., $\theta = 0$, i.e., the vectors must be parallel. Example 2(a) clearly demonstrates that the cross product is not associative. Also, it is not commutative; changing the multiplication order changes the sign, i.e., $\underline{u} \times \underline{v} = -\underline{v} \times \underline{u}$. However, the distributive property does apply for cross products.

4. The equation of a plane. In Section 13.4, when you were given three points in a plane, you had to solve three simultaneous equations. The three points not lying on a line define two vectors in the plane. Their cross product is perpendicular to both vectors, and therefore, it is normal to the plane. If this normal is $\underline{n} = A\underline{i} + B\underline{j} + C\underline{k}$, the plane is $Ax + By + Cz + D = 0$. You get D by substituting one of the points.

SOLUTIONS TO EVERY OTHER ODD EXERCISE

1. Use the algebraic rules for dot products along with the following: $\underline{i}\times\underline{i}=\underline{0}$, $\underline{i}\times\underline{j}=\underline{k}$, $\underline{i}\times\underline{k}=-\underline{j}$, $\underline{j}\times\underline{i}=-\underline{k}$, $\underline{j}\times\underline{j}=\underline{0}$, $\underline{j}\times\underline{k}=\underline{i}$, $\underline{k}\times\underline{i}=\underline{j}$, $\underline{k}\times\underline{j}=-\underline{i}$, and $\underline{k}\times\underline{k}=\underline{0}$. Thus, $(\underline{i}-\underline{j}+\underline{k}) \times (\underline{j}-\underline{k}) = \underline{i}\times\underline{j} - \underline{j}\times\underline{j} + \underline{k}\times\underline{j} - \underline{i}\times\underline{k} + \underline{j}\times\underline{k} - \underline{k}\times\underline{k} = \underline{k} - \underline{0} - \underline{i} + \underline{j} + \underline{i} - \underline{0} = \underline{j} + \underline{k}$.

5. Use the method of Exercise 1. $(3\underline{i} + 2\underline{j}) \times 3\underline{j} = 9(\underline{i} \times \underline{j}) + 6(\underline{j} \times \underline{j}) = 9\underline{k} + \underline{0}$. Thus, $[(3\underline{i} + 2\underline{j}) \times 3\underline{j}] \times (2\underline{i} - \underline{j} + \underline{k}) = 9\underline{k} \times (2\underline{i} - \underline{j} + \underline{k}) = 18(\underline{k} \times \underline{i}) - 9(\underline{k} \times \underline{j}) + 9(\underline{k} \times \underline{k}) = 9\underline{i} + 18\underline{j}$.

9. Use the method of Exercise 1 to get $(\underline{i} + \underline{k}) \times (\underline{i} + \underline{j} + \underline{k}) = \underline{i} \times \underline{i} + \underline{k} \times \underline{i} + \underline{i} \times \underline{j} + \underline{k} \times \underline{j} + \underline{i} \times \underline{k} + \underline{k} \times \underline{k} = \underline{0} + \underline{j} + \underline{k} - \underline{i} - \underline{j} + \underline{0} = -\underline{i} + \underline{k}$.

13. The area of the parallelogram spanned by $\underline{v}_1$ and $\underline{v}_2$ is $\|\underline{v}_1 \times \underline{v}_2\|$. In this case, $\underline{i} \times (\underline{i} - 2\underline{j}) = \underline{i} \times \underline{i} - 2\underline{i} \times \underline{j} = \underline{0} - 2\underline{k} = -2\underline{k}$. Thus, the area is $\|-2\underline{k}\| = 2$.

17. An orthogonal unit vector is $\underline{v}_1 \times \underline{v}_2/\|\underline{v}_1 \times \underline{v}_2\|$. $(\underline{i} - \underline{j} - \underline{k}) \times (2\underline{i} - 2\underline{j} + \underline{k}) = 2\underline{i} \times \underline{i} - 2\underline{j} \times \underline{i} - 2\underline{k} \times \underline{i} - 2\underline{i} \times \underline{j} + 2\underline{j} \times \underline{j} + 2\underline{k} \times \underline{j} + \underline{i} \times \underline{k} - \underline{j} \times \underline{k} - \underline{k} \times \underline{k} = \underline{0} + 2\underline{k} - 2\underline{j} - 2\underline{k} + \underline{0} - 2\underline{i} - \underline{j} - \underline{i} - \underline{0} = -3\underline{i} - 3\underline{j}$ and $\|-3\underline{i} - 3\underline{j}\| = \sqrt{9 + 9 + 0} = 3\sqrt{2}$. Thus, the desired vector is $(-1/\sqrt{2})\underline{i} - (1/\sqrt{2})\underline{j}$.

21. $2\underline{i} - \underline{k}$ and $4\underline{j} - 3\underline{k}$ are vectors in the desired plane. $(2\underline{i} - \underline{k}) \times (4\underline{j} - 3\underline{k})$ is normal to the plane, and it equals $8\underline{i} \times \underline{j} - 4\underline{k} \times \underline{j} - 6\underline{i} \times \underline{k} + 3\underline{k} \times \underline{k} = 8\underline{k} + 4\underline{i} + 6\underline{j} + \underline{0}$. The plane has the form $4x + 6y + 8z + D = 0$. Substituting $(0,0,0)$ yields $D = 0$, so the plane is $4x + 6y + 8z = 0$, i.e., $2x + 3y + 4z = 0$.

25. The area of the triangle is half of the area of the parallelogram which has the same vertices. The vector from $(0,1,2)$ to $(3,4,5)$ is $\underline{v}_1 = 3\underline{i} + 3\underline{j} + 3\underline{k}$, and the vector from $(0,1,2)$ to $(-1,-1,0)$ is $\underline{v}_2 = -\underline{i} - 2\underline{j} - 2\underline{k}$. $\underline{v}_1 \times \underline{v}_2 = -3\underline{i} \times \underline{i} - 3\underline{j} \times \underline{i} - 3\underline{k} \times \underline{i} - 6\underline{i} \times \underline{j} - 6\underline{j} \times \underline{j} - 6\underline{k} \times \underline{j} - 6\underline{i} \times \underline{k} - 6\underline{j} \times \underline{k} - 6\underline{k} \times \underline{k} = \underline{0} + 3\underline{k} - 3\underline{j} - 6\underline{k} - \underline{0} + 6\underline{i} + 6\underline{j} - 6\underline{i} - \underline{0} = 3\underline{j} - 3\underline{k}$. $\|\underline{v}_1 \times \underline{v}_2\| = \sqrt{0 + 9 + 9} = 3\sqrt{2}$, so the area of the triangle is $3\sqrt{2}/2$.

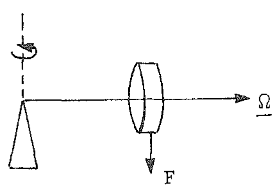
29. Let $\underline{v}_1 = b_1\underline{i} + c_1\underline{j} + d_1\underline{k}$ and $\underline{v}_2 = b_2\underline{i} + c_2\underline{j} + d_2\underline{k}$. By the component formula, $(a\underline{v}_1) \times \underline{v}_2 = (ac_1d_2 - ad_1c_2)\underline{i} + (ad_1b_2 - ab_1d_2)\underline{j} + (ab_1c_2 - ac_1b_2)\underline{k}$. The right-hand side is $a(\underline{v}_1 \times \underline{v}_2) = a[(c_1d_2 - d_1c_2)\underline{i} + (d_1b_2 - b_1d_2)\underline{j} + (b_1c_2 - c_1b_2)\underline{k}] = (ac_1d_2 - ad_1c_2)\underline{i} + (ad_1b_2 - ab_1d_2)\underline{j} + (ab_1c_2 - ac_1b_2)\underline{k}$.

33. Apply the result of Exercise 32. The direction $\underline{d}$ of the line is $-b\underline{i} + a\underline{j}$ and $(0,c/b)$ is a point P_0 on the line. Hence, $x\underline{i} + (y - c/b)\underline{j}$ is a vector from $(0,c/b)$ to (x,y). The distance is $\|(x\underline{i} + (y - c/b)\underline{j}) \times (-b\underline{i} + a\underline{j})\|/\|-b\underline{i} + a\underline{j}\| = \|-bx\underline{i} \times \underline{i} - b(y - c/b)\underline{j} \times \underline{i} + ax\underline{i} \times \underline{j} + a(y - c/b)\underline{j} \times \underline{j}\|/\sqrt{a^2 + b^2} = \|-(c - by)\underline{k} + ax\underline{k}\|/\sqrt{a^2 + b^2} = |ax + by - c|/\sqrt{a^2 + b^2}$.

37. Since all vectors in this problem are unit vectors, $\|\underline{N} \times \underline{a}\| = \sin\theta_1$ and $\|\underline{N} \times \underline{b}\| = \sin\theta_2$. From Snell's law, $n_1 \sin\theta_1 = n_2 \sin\theta_2$. Hence, $n_1\|\underline{N} \times \underline{a}\| = n_2\|\underline{N} \times \underline{b}\|$.

In order to establish that $\underline{N} \times \underline{a}$ and $\underline{N} \times \underline{b}$ have the same direction, we assume that $\underline{N}$, $\underline{a}$, and $\underline{b}$ all lie in the same plane, and $\underline{a}$ and $\underline{b}$ are on the same side of $\underline{N}$. Hence, $\underline{N} \times \underline{a}$ and $\underline{N} \times \underline{b}$ both are perpendicular to this plane and parallel to each other. Thus, $n_1\|\underline{N} \times \underline{a}\|$ and $n_2\|\underline{N} \times \underline{b}\|$ have the same direction, as well as the same magnitude, and so are equal.

41.



The direction in which the flywheel turns is $\underline{\Omega} \times \underline{F}$. As shown in the diagram, the gyroscope will rotate to the left (as viewed from above).

SECTION QUIZ

1. A parallelogram has vertices at $(5,2,1)$, $(0,-1,1)$, and $(3,-1,4)$. What is the area of the parallelogram?

2. $\underline{u} \times \underline{v} = \underline{0}$ has what geometric interpretation? Assume that neither $\underline{u}$ nor $\underline{v}$ are $\underline{0}$.

3. Use the cross product to find the plane containing $(3,1,1)$, $(2,3,0)$, and $(2,3,-1)$.

4. Compute the following cross products:

   (a) $(3\underline{i} + \underline{j}) \times \underline{k}$

   (b) $(2\underline{k}) \times (2\underline{i} - \underline{k})$

   (c) $(\underline{i} + \underline{j} + \underline{k}) \times (3\underline{i} + 2\underline{j} - 2\underline{k})$

5. A certain city, which was designated as the worst city in the country to live in, has been left unprotected by the Defense Department. Consequently, the city must provide its own defenses. Their missile neutralizer works only at a perpendicular to the base of the weapon. If, at a certain instant, two vectors along the base are $2\underline{i} + 3\underline{j}$ and $\underline{i} - \underline{j} + 2\underline{k}$, what is the line along which a nuclear warhead can be neutralized? The neutralizer originates at $(1,1,0)$.

ANSWERS TO PREREQUISITE QUIZ

1. $9x + 3y + 4z = 9$

2. Left-handed

ANSWERS TO SECTION QUIZ

1. $\sqrt{387}$

2. $\underline{u}$ and $\underline{v}$ are parallel.

3. $-2x - y + 7 = 0$

4. (a) $\underline{i} - 3\underline{j}$

   (b) $4\underline{j}$

   (c) $-4\underline{i} + 5\underline{j} - \underline{k}$

5. $(1,1,0) + (6,-4,-5)t$

13.6 Matrices and Determinants

PREREQUISITES

1. Recall how to compute a cross product (Section 13.5).

PREREQUISITE QUIZ

1. Compute $\underline{i} \times \underline{j}$.
2. Compute $(2\underline{i} - 3\underline{k}) \times (\underline{j} + \underline{k})$.

GOALS

1. Be able to compute 2×2 and 3×3 determinants.
2. Be able to use determinants for computing cross products.

STUDY HINTS

1. Determinants. The determinant of a matrix is a number whose absolute value is the area of the parallelogram or the volume of the parallelepiped spanned by a given set of 2 or 3 vectors. Note that the components of the vector are listed across a row. Also note that determinants always have a square configuration.
2. Orientation. A positive determinant tells you that two vectors which make up a determinant have a counterclockwise orientation or that the three vectors form a right-handed system.
3. Matrices. A matrix is simply a rectangular (not necessarily square) array of numbers. It is not a number. The usefulness of matrices will be seen in Chapter 15. Two matrices are equal if and only if corresponding components are equal. Compare this with determinants which may be equal even though the components are not equal.

4. Notation. Determinants are represented by numbers within vertical bars. Matrices are numbers within brackets.

5. Computing 2×2 determinants. You should memorize the formula $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$. Just take the product of the diagonal going left to right and subtract the product of the diagonal going in the opposite direction.

6. Computing 3×3 determinants. Use the checkerboard pattern shown on p. 687 which begins with a plus sign in the upper left corner. Choose any column or row — usually picking the one with the most zeros saves work. Draw vertical and horizontal lines through the first number of the row or column. The numbers remaining form a 2×2 determinant, which should be multiplied by the number (with sign determined by the checkerboard) through which both lines were drawn. Repeat for the remaining numbers of the row or column. Finally, sum the results. This process, called expansion by minors, works for any row or column . Be sure to use the correct sign.

7. Cross products and determinants. The cross product can be computed as a 3×3 determinant. This is the easiest way to remember how to compute the cross product. The determinant is written with $\underline{i}$, $\underline{j}$, and $\underline{k}$ across the top row. The components of the first vector is written across the middle row and the components of the second vector is on the bottom row. See Example 5(b).

8. Triple products. The volume of the parallelepiped spanned by $\underline{v}_1$, $\underline{v}_2$, and $\underline{v}_3$ may be computed using the triple product $(\underline{v}_1 \times \underline{v}_2) \cdot \underline{v}_3$. Example 8 demonstrates an interesting fact about triple products.

SOLUTIONS TO EVERY OTHER ODD EXERCISE

1. Using the fact that $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$, we have $\begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} = (1)(1) - (-1)(1) = 2$.

5. Since $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$, we get $\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = (1)(4) - (2)(3) = -2$.

9. Since $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$, we get $\begin{vmatrix} a & b \\ 0 & c \end{vmatrix} = ac = b(0) = ac$.

13. The left-hand side is $\begin{vmatrix} ra & rb \\ c & d \end{vmatrix} = rad - rbc$ and the right-hand side is $r\begin{vmatrix} a & b \\ c & d \end{vmatrix} = r(ad - bc) = rad - rbc$. Therefore, the identity is proven.

17. Expand in minors of the first row. $\begin{vmatrix} 2 & -1 & 0 \\ 4 & 3 & 2 \\ 3 & 0 & 1 \end{vmatrix} = 2\begin{vmatrix} 3 & 2 \\ 0 & 1 \end{vmatrix} + 1\begin{vmatrix} 4 & 2 \\ 3 & 1 \end{vmatrix} + 0\begin{vmatrix} 4 & 3 \\ 3 & 0 \end{vmatrix} = 2(3) - 2 + 0 = 4$.

21. Expand in minors of the first row to get $\begin{vmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 1 & 0 & 2 \end{vmatrix} = 2\begin{vmatrix} 2 & 1 \\ 0 & 2 \end{vmatrix} - 1\begin{vmatrix} 0 & 1 \\ 1 & 2 \end{vmatrix} + 0\begin{vmatrix} 0 & 2 \\ 1 & 0 \end{vmatrix} = 2(4) - 1(-1) + 0 = 9$.

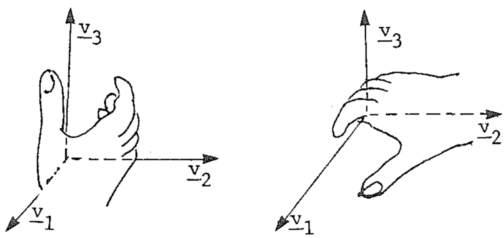
25. In determinant form, $(a_1\underline{i} + b_1\underline{j} + c_1\underline{k}) \times (a_2\underline{i} + b_2\underline{j} + c_2\underline{k})$ is $\begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}$. Thus, $(3\underline{i} - \underline{j}) \times (\underline{j} + \underline{k}) = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 3 & -1 & 0 \\ 0 & 1 & 1 \end{vmatrix} = \begin{vmatrix} -1 & 0 \\ 1 & 1 \end{vmatrix}\underline{i} - \begin{vmatrix} 3 & 0 \\ 0 & 1 \end{vmatrix}\underline{j} + \begin{vmatrix} 3 & -1 \\ 0 & 1 \end{vmatrix}\underline{k} = -\underline{i} - 3\underline{j} + 3\underline{k}$.

29. By the method of Exercise 25, we get $(\underline{i} - \underline{k}) \times (\underline{i} + \underline{k}) = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 0 & -1 \\ 1 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 0 & -1 \\ 0 & 1 \end{vmatrix}\underline{i} - \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix}\underline{j} + \begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix}\underline{k} = -2\underline{j}$.

33. The volume of the parallelepiped spanned by $a_1\underline{i} + a_2\underline{j} + a_3\underline{k}$, $b_1\underline{i} + b_2\underline{j} + b_3\underline{k}$, and $c_1\underline{i} + c_2\underline{j} + c_3\underline{k}$, is the absolute value of $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$. The vectors from $(1,1,2)$ to $(2,0,2)$, $(3,1,3)$, and $(2,2,-3)$ are $\underline{i} - \underline{j}$, $2\underline{i} + \underline{k}$, and $\underline{i} + \underline{j} - 5\underline{k}$, respectively. Thus, $\begin{vmatrix} 1 & -1 & 0 \\ 2 & 0 & 1 \\ 1 & 1 & -5 \end{vmatrix} = 1\begin{vmatrix} 0 & 1 \\ 1 & -5 \end{vmatrix} + 1\begin{vmatrix} 2 & 1 \\ 1 & -5 \end{vmatrix} + 0\begin{vmatrix} 2 & 0 \\ 1 & 1 \end{vmatrix} = -1 - 11 + 0 = -12$, and

so the volume is 12 .

37. Consider $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$. It is $a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix} = a(ei - fh) -$ $b(di - fg) + c(dh - eg)$. Now, interchange the first two columns and the determinant becomes $b\begin{vmatrix} d & f \\ g & i \end{vmatrix} - a\begin{vmatrix} e & f \\ h & i \end{vmatrix} + c\begin{vmatrix} e & d \\ h & g \end{vmatrix} = b(di - fg) -$ $a(ei - fh) + c(eg - dh)$, as required.

41. 

Pick an arbitrary right-handed set of vectors and apply the right-hand rule.

45. If $x = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}/D$, $y = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}/D$, and $z = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}/D$, then $a_1x + b_1y + c_1z = (1/D)[a_1(b_2c_3d_1 + b_1c_2d_3 + b_3c_1d_2 - b_2c_1d_3 -$ $b_3c_2d_1 - b_1c_3d_2) + b_1(a_1c_3d_2 + a_3c_2d_1 + a_2c_1d_3 - a_3c_1d_2 - a_1c_2d_3 -$ $a_2c_3d_1) + c_1(a_1b_2d_3 + a_3b_1d_2 + a_2b_3d_1 - a_3b_2d_1 - a_1b_3d_2 - a_2b_1d_3)] =$ $(d_1/D)(a_1b_2c_3 - a_1b_3c_2 + a_3b_1c_2 - a_2b_1c_3 + a_2b_3c_1 - a_3b_2c_1) = d_1$. Similarly, we can show that $a_2x + b_2y + c_2z = d_2$ and $a_3x + b_3y +$ $c_3z = d_3$.

49. Expanding along the first column, we get $\begin{vmatrix} a & d & g \\ b & e & h \\ c & g & i \end{vmatrix} = a\begin{vmatrix} e & h \\ f & i \end{vmatrix} - b\begin{vmatrix} d & g \\ f & i \end{vmatrix} +$ $c\begin{vmatrix} d & g \\ e & h \end{vmatrix}$. Expanding along the first row, we get $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} -$ $b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$, which is $a\begin{vmatrix} e & h \\ f & i \end{vmatrix} - b\begin{vmatrix} d & g \\ f & i \end{vmatrix} + c\begin{vmatrix} d & g \\ e & h \end{vmatrix}$, by the result of Example 2(b). Thus, the determinant of the transpose of a 3×3 matrix is equal to the determinant of the original matrix.

53. Since all of the components of the third row of Exercise 18 are all zero except for one, we expand along that row. This yields $\begin{vmatrix} 1 & 1 & 2 \\ 2 & -1 & 1 \\ 1 & 0 & 0 \end{vmatrix} =$ $1\begin{vmatrix} 1 & 2 \\ -1 & 1 \end{vmatrix} + 0 + 0 = 1 + 2 = 3$.

For Exercise 19, we apply the result of Exercise 50. The objective is to obtain a determinant with one row or column whose entries are all zero except for one. Add the third column to the second column to get

$$\begin{vmatrix} 1 & 2 & 3 \\ -1 & -1 & 2 \\ 0 & 1 & -1 \end{vmatrix} = \begin{vmatrix} 1 & 5 & 3 \\ -1 & 1 & 2 \\ 0 & 0 & -1 \end{vmatrix} = 0 + 0 - 1\begin{vmatrix} 1 & 5 \\ -1 & 1 \end{vmatrix} = -1(1 + 5) = -6 .$$

SECTION QUIZ

1. (a) Compute $\begin{vmatrix} 2 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 3 & 2 \end{vmatrix}$. What is the geometric interpretation of this determinant? Is this a right-handed system?

(b) Compute $\begin{vmatrix} -1 & 3 & 2 \\ 2 & -6 & -4 \\ 3 & 0 & 1 \end{vmatrix}$. What is the geometric interpretation of this determinant?

2. $\begin{vmatrix} 5 & -3 & 1 \\ 2 & 0 & 2 \\ 8 & -1 & 3 \end{vmatrix} = 5\begin{vmatrix} 0 & 2 \\ -1 & 3 \end{vmatrix} - 3\begin{vmatrix} 2 & 2 \\ 8 & 3 \end{vmatrix} + 1\begin{vmatrix} 2 & 0 \\ 8 & -1 \end{vmatrix} = 5(-2) - 3(-10) + 1(-2) = 18$.

Two errors were made in the calculation. What are they?

3. Is the determinant in Question 2 equal to $-2\begin{vmatrix} -3 & 1 \\ -1 & 3 \end{vmatrix} - 2\begin{vmatrix} 5 & -3 \\ 8 & -1 \end{vmatrix}$? Explain why or why not.

4. Compute the following cross products:

   (a) $(3\underline{i} + 3\underline{j}) \times (\underline{i} + 2\underline{j})$

   (b) $(\underline{i} + \underline{j}) \times (-\underline{k})$

   (c) $(2\underline{i} + 4\underline{j} + 6\underline{k}) \times (\underline{i} + 2\underline{j} + 3\underline{k})$

5. Having found a key to the executive washroom, you decide to find out why the executive washroom is so special. Inside, you find silkworms busily manufacturing silk towellettes to fill a crystal parallelepiped. Vertices of the parallelepiped located adjacent to $(0,1,1)$ are $(1,3,2)$, $(2,5,4)$, and $(-1,-3,0)$. Use determinants to compute what volume of silk is necessary to fill the parallelepiped crystal.

ANSWERS TO PREREQUISITE QUIZ

1. $\underline{k}$

2. $3\underline{i} - 2\underline{j} + 2\underline{k}$

ANSWERS TO SECTION QUIZ

1. (a) -1 ; the parallelepiped spanned by the vectors $(2,1,0)$, $(0,1,1)$, and $(1,3,2)$ has volume 1 ; it is a left-handed system.

   (b) 0 ; all three vectors lie in a single plane.

2. $-3\begin{vmatrix} 2 & 2 \\ 8 & 3 \end{vmatrix}$ should be $+3\begin{vmatrix} 2 & 2 \\ 8 & 3 \end{vmatrix}$; $\begin{vmatrix} 0 & 2 \\ -1 & 3 \end{vmatrix} = +2$, not -2 .

3. Yes; this is the expansion across the second row.

4. (a) $3\underline{k}$

   (b) $-\underline{i} + \underline{j}$

   (c) $\underline{0}$

5. 2

13.R Review Exercises for Chapter 13

SOLUTIONS TO EVERY OTHER ODD EXERCISE

1. By the addition of ordered pairs, $(3,2) + (-1,6) = (3 - 1,2 + 6) = (2,8)$.

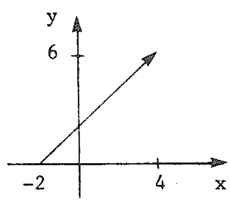
5. In terms of ordered triples, $3\underline{i} + 2\underline{j}$ corresponds to $(3,2,0)$ and $8\underline{i} - \underline{j} - \underline{k}$ corresponds to $(8,-1,-1)$. Thus, $(3,2,0) + (8,-1,-1) = (11,1,-1)$ corresponds to $11\underline{i} + \underline{j} - \underline{k}$.

9. By the definition of dot products, $(a_1\underline{i} + b_1\underline{j} + c_1\underline{k})\cdot(a_2\underline{i} + b_2\underline{j} + c_2\underline{k}) = a_1a_2 + b_1b_2 + c_1c_2$. Therefore, $(8\underline{i} + 3\underline{j} - \underline{k})\cdot(\underline{i} - \underline{j} - \underline{k}) = 8(1) + 3(-1) + (-1)(-1) = 6$.

13. $\underline{u} \times \underline{v} = (2\underline{i} + \underline{j}) \times \underline{k} = 2\underline{i} \times \underline{k} + \underline{j} \times \underline{k}$. Using the fact that $\underline{i} \times \underline{k} = -\underline{j}$ and $\underline{j} \times \underline{k} = \underline{i}$, the cross product becomes $\underline{i} - 2\underline{j}$.

17. Since $\underline{v}_1 \times \underline{v}_2$ is orthogonal to both $\underline{v}_1$ and $\underline{v}_2$, a vector orthogonal to $3\underline{i} + 2\underline{k}$ and $\underline{j} - \underline{k}$ is $(3\underline{i} + 2\underline{k}) \times (\underline{j} - \underline{k}) = 3\underline{i} \times \underline{j} + 2\underline{k} \times \underline{j} - 3\underline{i} \times \underline{k} - 2\underline{k} \times \underline{k} = 3\underline{k} - 2\underline{i} + 3\underline{j} - \underline{0}$. Its length is $\sqrt{9 + 4 + 9} = \sqrt{22}$. Thus, $\underline{u} = (-2/\sqrt{22})\underline{i} + (3/\sqrt{22})\underline{j} + (3/\sqrt{22})\underline{k}$.

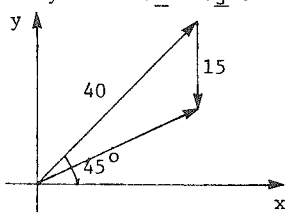
21. (a)



The vector joining $(-2,0)$ to $(4,6)$ has components $(4 + 2,6 - 0) = (6,6)$.

(b) The vector joining $(-2,0)$ to $(1,1)$ is $3\underline{i} + \underline{j}$. Adding $\underline{v}$ yields $9\underline{i} + 7\underline{j}$.

25.



Geometrically, the bird's velocity vector is 40 at 45° is $(40/\sqrt{2})\underline{i} + (40/\sqrt{2})\underline{j}$. The wind's velocity vector is $-15\underline{j}$. The speed of the bird relative to the earth's surface is determined by the sum of the two vectors: $(40/\sqrt{2})\underline{i} + (40/\sqrt{2} - 15)\underline{j}$. The speed is the length: $(800 + 800 - 1200/\sqrt{2} + 225)^{1/2} = (1825 - 600\sqrt{2})^{1/2} \approx 31.25$ km/hr.

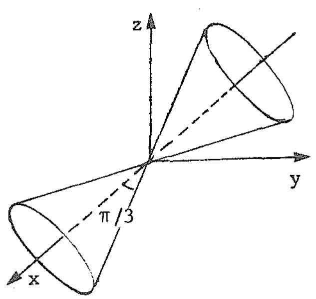
29. The direction of the line is $\underline{i} + \underline{j} + \underline{k}$, so it is $x = 1 + t$, $y = 1 + t$, $z = 2 + t$.

33. The vectors from $(1,1,2)$ to $(2,2,3)$ and $(0,0,0)$ are $\underline{i} + \underline{j} + \underline{k}$ and $-\underline{i} - \underline{j} - 2\underline{k}$, respectively. The normal to the plane is given by the cross product: $\begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 1 & 1 \\ -1 & -1 & -2 \end{vmatrix} = -\underline{i} + \underline{j}$. Thus, the plane is $-x + y + D = 0$. Substituting $(0,0,0)$ yields $D = 0$, so the plane is $-x + y = 0$.

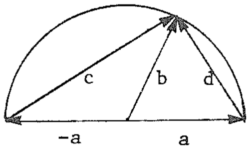
37. The orthogonal line has direction $-\underline{i} + \underline{j}$, obtained from the coefficients of x, y, and z. Thus, the line is $x = -t$, $y = t$, $z = 3$.

41. From the coefficients, the orthogonal vector has direction $\underline{i} - 6\underline{j} + \underline{k}$. Its length is $\sqrt{1 + 36 + 1} = \sqrt{38}$. Thus, the desired unit vector is $(1/\sqrt{38})\underline{i} - (6/\sqrt{38})\underline{j} + (1/\sqrt{38})\underline{k}$.

45. Let the desired vector be $\underline{u} = a\underline{i} + b\underline{j} + c\underline{k}$. Then, $\underline{u}\cdot\underline{i} = \cos 30^\circ = \sqrt{3}/2 = a$. Also, $\underline{u}\cdot\underline{j} = b = \underline{u}\cdot\underline{k} = c$. Since $\|\underline{u}\| = 1$, $a^2 + b^2 + c^2 = 1$. Substitute for a and b to get $3/4 + 2c^2 = 1$, so $c = 1/2\sqrt{2}$. Therefore, $\underline{u} = (\sqrt{3}/2)\underline{i} + (1/2\sqrt{2})\underline{j} + (1/2\sqrt{2})\underline{k}$.

49. 

This is a (double) cone with its vertex at the origin, its axis along the x-axis, and an apical angle of 60°.

53. 

From the diagram, we see that $-\mathbf{a} + \mathbf{c} = \mathbf{b}$ and $\mathbf{a} + \mathbf{d} = \mathbf{b}$. Thus, $(-\mathbf{a} + \mathbf{c})\cdot(\mathbf{a} + \mathbf{d}) = \mathbf{b}\cdot\mathbf{b} = \|\mathbf{b}\|^2 = -\mathbf{a}\cdot\mathbf{a} + \mathbf{c}\cdot\mathbf{a} - \mathbf{a}\cdot\mathbf{d} + \mathbf{c}\cdot\mathbf{d} = -\|\mathbf{a}\|^2 + \mathbf{a}\cdot(\mathbf{c} - \mathbf{d}) + \mathbf{c}\cdot\mathbf{d}$. Since $\|\mathbf{b}\| = \|\mathbf{a}\|$, we have $2\|\mathbf{a}\|^2 = \mathbf{a}\cdot(\mathbf{c} - \mathbf{d}) + \mathbf{c}\cdot\mathbf{d}$. Now, $\mathbf{c} - \mathbf{d} = 2\mathbf{a}$, so $2\|\mathbf{a}\|^2 = \mathbf{a}\cdot(2\mathbf{a}) + \mathbf{c}\cdot\mathbf{d} = 2\|\mathbf{a}\|^2 + \mathbf{c}\cdot\mathbf{d}$, i.e., $\mathbf{c}\cdot\mathbf{d} = 0$. Therefore $\mathbf{c}$ is perpendicular to $\mathbf{d}$.

57. The determinant $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ is $ad - bc$, so $\begin{vmatrix} -1 & -1 \\ 2 & 1 \end{vmatrix} = (-1)(1) - (-1)(2) = -1 + 2 = 1$.

61. Expanding across the first row yields $\begin{vmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{vmatrix} = 1\begin{vmatrix} 2 & 2 \\ 3 & 3 \end{vmatrix} - 1\begin{vmatrix} 2 & 2 \\ 3 & 3 \end{vmatrix} + 1\begin{vmatrix} 2 & 2 \\ 3 & 3 \end{vmatrix} = 0$.

65. The volume is the absolute value of $\begin{vmatrix} 1 & -1 & -1 \\ 2 & 1 & -5 \\ 8/3 & -1 & 1/2 \end{vmatrix} = 1\begin{vmatrix} 1 & -5 \\ -1 & 1/2 \end{vmatrix} + 1\begin{vmatrix} 2 & -5 \\ 8/3 & 1/2 \end{vmatrix} - 1\begin{vmatrix} 2 & 1 \\ 8/3 & -1 \end{vmatrix} = -9/2 + 40/3 + 14/3 = 29/2$.

69. $\|\mathbf{r} - \mathbf{r}_i\|^2 = (\mathbf{r} - \mathbf{r}_i)\cdot(\mathbf{r} - \mathbf{r}_i) = ((\mathbf{r} - \mathbf{c}) + (\mathbf{c} - \mathbf{r}_i))\cdot((\mathbf{r} - \mathbf{c}) + (\mathbf{c} - \mathbf{r}_i)) = (\mathbf{r} - \mathbf{c})\cdot(\mathbf{r} - \mathbf{c}) + 2(\mathbf{c} - \mathbf{r}_i)\cdot(\mathbf{r} - \mathbf{c}) + (\mathbf{c} - \mathbf{r}_i)\cdot(\mathbf{c} - \mathbf{r}_i) = \|\mathbf{r} - \mathbf{c}\|^2 + 2(\mathbf{c} - \mathbf{r}_i)\cdot(\mathbf{r} - \mathbf{c}) + \|\mathbf{c} - \mathbf{r}_i\|^2$. Therefore, $S = \Sigma_{i=1}^n m_i\|\mathbf{r} - \mathbf{r}_i\|^2 = \Sigma_{i=1}^n m_i\{\|\mathbf{r} - \mathbf{c}\|^2 + 2(\mathbf{c} - \mathbf{r}_i)\cdot(\mathbf{r} - \mathbf{c}) + \|\mathbf{r}_i - \mathbf{c}\|^2\} = \|\mathbf{r} - \mathbf{c}\|^2\Sigma_{i=1}^n m_i + 2\Sigma_{i=1}^n m_i[\mathbf{c}\cdot(\mathbf{r} - \mathbf{c}) - \mathbf{r}_i\cdot(\mathbf{r} - \mathbf{c})] + \Sigma_{i=1}^n m_i\|\mathbf{r}_i - \mathbf{c}\|^2 = \Sigma_{i=1}^n m_i\|\mathbf{r}_i - \mathbf{c}\|^2 + m\|\mathbf{r} - \mathbf{c}\|^2 + 2\mathbf{c}\cdot(\mathbf{r} - \mathbf{c})\Sigma_{i=1}^n m_i - 2\Sigma_{i=1}^n m_i\mathbf{r}_i\cdot(\mathbf{r} - \mathbf{c})$. The last two terms become $2\mathbf{c}\cdot(\mathbf{r} - \mathbf{c})m - 2(\mathbf{r} - \mathbf{c})\cdot\Sigma_{i=1}^n m_i\mathbf{r}_i = 2\mathbf{c}\cdot(\mathbf{r} - \mathbf{c})m - 2(\mathbf{r} - \mathbf{c})\cdot m\mathbf{c} = 0$. Therefore, $S = \Sigma_{i=1}^n m_i\|\mathbf{r}_i - \mathbf{c}\|^2 + m\|\mathbf{r} - \mathbf{c}\|^2$.

73. The objective is to obtain a row or column whose entries are all zero except for one. This is done by adding a multiple of one row or column to another. In this case, we add row 2 to row 1 and then add -1/2 of column 1 to column 2. This yields $\begin{vmatrix} 6 & 2 & -3 \\ 2 & 2 & 3 \\ 4 & 8 & -1 \end{vmatrix} = \begin{vmatrix} 8 & 4 & 0 \\ 2 & 2 & 3 \\ 4 & 8 & -1 \end{vmatrix} =$ $\begin{vmatrix} 8 & 0 & 0 \\ 2 & 1 & 3 \\ 4 & 6 & -1 \end{vmatrix} = 8\begin{vmatrix} 1 & 3 \\ 6 & -1 \end{vmatrix} = 8(-1 - 18) = -162$. We expanded along the first row.

77. By Example 8, Section 13.6, $(\underline{v} \times \underline{j})\cdot\underline{k} = (\underline{j} \times \underline{k})\cdot\underline{v} = \underline{i}\cdot\underline{v} = 0$. Thus, $\underline{v}$ is orthogonal to $\underline{i}$.

81. Let $\underline{u} = a_1\underline{i} + b_1\underline{j} + c_1\underline{k}$, $\underline{v} = a_2\underline{i} + b_2\underline{j} + c_2\underline{k}$, and $\underline{w} = a_3\underline{i} + b_3\underline{j} + c_3\underline{k} \neq \underline{0}$.

(a) $\underline{v}\cdot\underline{w} = \|\underline{v}\|\|\underline{w}\| \cos\theta$ by definition. Since $\|\underline{w}\| \neq 0$ and there exist values of θ such that $\cos\theta \neq 0$, then $\underline{v}\cdot\underline{w} = 0$ for all $\underline{w}$. This implies that $\|\underline{v}\| = 0 = \sqrt{a_2^2 + b_2^2 + c_2^2}$. Therefore, $a_2 = b_2 = c_2 = 0$, and $\underline{v} = \underline{0}$.

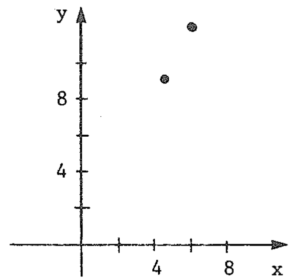
(b) $\underline{u}\cdot\underline{w} = \underline{v}\cdot\underline{w}$ implies $\underline{u}\cdot\underline{w} - \underline{v}\cdot\underline{w} = 0$. Therefore, $(\underline{u} - \underline{v})\cdot\underline{w} = 0$. From part (a), we conclude that $\underline{u} - \underline{v} = \underline{0}$, so $\underline{u} = \underline{v}$.

(c) We have $\underline{v}\cdot\underline{i} = a_2 = \underline{v}\cdot\underline{j} = b_2 = \underline{v}\cdot\underline{k} = c_2 = 0$, so $\underline{v} = \underline{0}$.

(d) We have $\underline{u}\cdot\underline{i} = a_1 = \underline{v}\cdot\underline{i} = a_2$; $\underline{u}\cdot\underline{j} = b_1 = \underline{v}\cdot\underline{j} = b_2$; and $\underline{u}\cdot\underline{k} = c_1 = \underline{v}\cdot\underline{k} = c_2$. Since their components are equal, $\underline{u} = \underline{v}$.

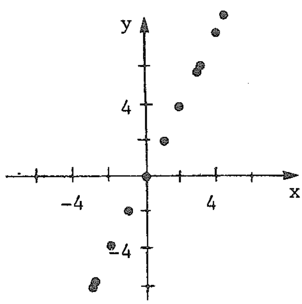
85. In each case, $rP + sQ = r(1,2) + s(\pi,2\pi) = (r + \pi s, 2r + 2\pi s)$. Therefore, all such points must lie on the line $y = 2x$.

(a)



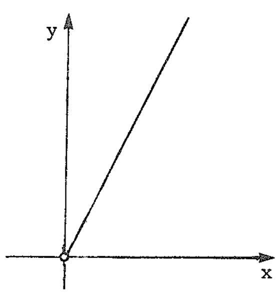
Here, the x-coordinate is $x = r + \pi s$, where r and s are any combination of positive integers. Thus, x is in the set $\{1 + \pi , 1 + 2\pi , 1 + 3\pi , \ldots , 2 + \pi , 2 + 2\pi , 2 + 3\pi , \ldots , 3 + \pi , 3 + 2\pi , 3 + 3\pi , \ldots \}$.

(b)



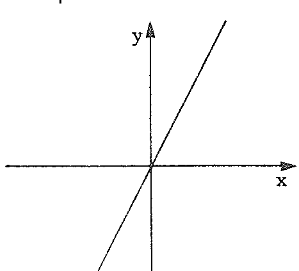
Here, the x-coordinate is $x = r + \pi s$, where r and s are any combination of integers. Thus, x is in the set $\{0 , \pm\pi , \pm 2\pi , \pm 3\pi , \ldots , \pm 1 , \pm 1 \pm \pi, \pm 1 \pm 2\pi , \pm 1 \pm 3\pi , \ldots , \pm 2 , \pm 2 \pm \pi , \pm 2 \pm 2\pi , \pm 2 \pm 3\pi , \ldots , \pm 3 , \pm 3 \pm \pi , \pm 3 \pm 2\pi , \pm 3 \pm 3\pi , \ldots\}$.

(c)



Here, the x-coordinate is $x = r + \pi s$, where r and s are any combination of positive real numbers. Thus, x is any positive real number.

(d)



Here, the x-coordinate is $x = r + \pi s$, where r and s are any combination of real numbers. Thus, x is any real number.

TEST FOR CHAPTER 13

1. True or false.

(a) Any non-zero vector $\underline{v}$, dotted with the zero vector, is the zero vector.

(b) Two vectors originating from different points in space are equal if both have the same magnitude and the same direction.

(c) Lines in space which do not lie in parallel planes must intersect.

(d) There is exactly one vector in space which can not be normalized to a unit vector.

(e) Both the cross product and the dot product may be used to calculate the angle between two vectors.

2. A plane P contains the origin and has a normal vector $2\underline{i} + \underline{j} - \underline{k}$. Find a plane perpendicular to P which contains the line $x = 5 + 2t$, $y = 3 - t$, $z = -10 + 5t$.

3. Perform the following calculations:

(a) $(\underline{i} + \underline{j} - \underline{k}) \cdot (2\underline{i} - 3\underline{j})$

(b) $3(2,1) + (2,0)$

(c) $(2\underline{i} - \underline{k}) \times (\underline{i} + \underline{j})$

(d) $\begin{vmatrix} 2 & 0 & 1 \\ 1 & 0 & 5 \\ 0 & -1 & 3 \end{vmatrix}$

4. (a) State how the cross product $\underline{u} \times \underline{v}$ is related to the angle θ between $\underline{u}$ and $\underline{v}$.

(b) State how the dot product $\underline{u} \cdot \underline{v}$ is related to the angle θ between $\underline{u}$ and $\underline{v}$.

(c) Find a unit vector in the plane which is orthogonal to $3\underline{i} - 2\underline{j}$.

5. Let $(1,1,1)$, $(2,1,0)$, and $(1,-1,3)$ be the vertices of a triangle.

   (a) What is the area of the triangle?

   (b) What is the equation of the plane passing through the three points?

6. Find a set of parametric equations which describe the line passing through the point $(2,6,7)$ and perpendicular to the plane $4x + 3y - 4z + 2 = 0$.

7. Let L be the line $(-2,0,1) + t(3,2,2)$ and let M be the line $(3,5,4) + t(1,-1,1)$.

   (a) Where do the two lines intersect?

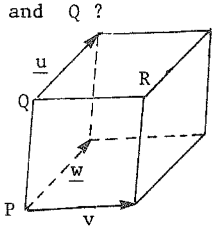
   (b) Find the equation of a plane which is parallel to the two lines and exactly one unit distance away from the lines.

8. Let P , Q , and R be the points $(1,0,0)$, $(0,2,3)$, and $(-1,0,-1)$, respectively.

   (a) Does $\overrightarrow{PQ} - \overrightarrow{PR}$ equal $\overrightarrow{QR}$ or $\overrightarrow{RQ}$?

   (b) What is the orthogonal projection of $\overrightarrow{PQ}$ upon $\overrightarrow{PR}$?

   (c) What is the distance from R to the line passing through P and Q ?

9. 

In the parallelepiped shown at the left, $\underline{u} = -2\underline{i} + \underline{j} + 2\underline{k}$, $\underline{v} = 3\underline{j} - \underline{k}$, $P = (0,0,0)$, and $Q = (1,1,1)$. The parallelepiped is not drawn to scale.

   (a) What are the coordinates of R ?

   (b) What is the distance from R to the plane spanned by $\underline{v}$ and $\underline{w}$?

   (c) Compute the volume of the parallelepiped.

10. On a brisk first day of Spring, a young man has just finished waxing his car for his date later that evening. Unfortunately, a bird dropping comes out of the sky from $(-2,0,10)$ and falls with velocity $-2\underline{k}$. A strong gust of wind suddenly begins blowing with velocity $\underline{i}$. If the corners of the car are located at $(2.5 \pm 0.3, 0 \pm 0.1, 0.05 \pm 0.05)$ and the young man is standing at $(2.9,0,0)$, will the bird dropping ruin the wax job? Explain your answer.

ANSWERS TO CHAPTER TEST

1. (a) False; dot products are scalars.
   (b) True
   (c) False; consider the x-axis in the xy-plane and any line in the plane $x = 1$ which does not intersect the x-axis.
   (d) True; the zero vector.
   (e) True
2. $-x + 3y + z + 6 = 0$
3. (a) -1
   (b) $(8,3)$
   (c) $\underline{i} - \underline{j} + 2\underline{k}$
   (d) 9
4. (a) $\|\underline{u} \times \underline{v}\| = \|\underline{u}\|\|\underline{v}\| \sin\theta$
   (b) $\underline{u}\cdot\underline{v} = \|\underline{u}\|\|\underline{v}\| \cos\theta$
   (c) $(2\underline{i} + 3\underline{j})\sqrt{13}$ or $(-2\underline{i} - 3\underline{j})/\sqrt{13}$
5. (a) $\sqrt{3}$
   (b) $-x - y - z + 3 = 0$

6. $x = 2 + 4t$, $y = 6 + 3t$, $z = 7 - 4t$

7. (a) $(4,4,5)$

   (b) $4x - y - 5z = -13 + \sqrt{42}$ or $4x - y - 5z = -13 - \sqrt{42}$

8. (a) $\overrightarrow{RQ}$

   (b) $(2\underline{i} - \underline{k})/5$

   (c) $\sqrt{69/14}$

9. (a) $(1,4,0)$

   (b) $15/\sqrt{89}$

   (c) 15

10. No; it travels along the line $(-2,0,10) + t(1,0,-2)$, so it will not hit the car. (The young man will probably be hit in the forehead.)